Int J Theor Phys (2009) 48: 71–84
DOI 10.1007/s10773-008-9783-1

# Quantum Groups $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-Invariant Bosonic Gases: A Comparative Study

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Received: 31 March 2008 / Accepted: 16 June 2008 / Published online: 3 July 2008


**Abstract** We discuss the algebras, representations, and thermodynamics of quantum group bosonic gas models with two different symmetries: $GL_{p,q}(2)$ and $SU_{q_1/q_2}(2)$. We establish the nature of the basic numbers which follow from these $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariant bosonic algebras. The Fock space representations of both of these quantum group invariant bosonic oscillator algebras are analyzed. It is concisely shown that these two quantum group invariant bosonic particle gases have different algebraic and high-temperature thermo-statistical properties.

**Keywords** Deformed bosons and fermions · Quantum groups · Statistical thermodynamics

## 1 Introduction

Quantum groups and algebras are specific deformations of the classical Lie groups and Lie algebras with some deformation parameter $q$ (real or complex) [1–6]. They have found many applications in a wide spectrum of research in physics. For instance, there are some possible connections between one-parameter deformed quantum groups and generalized statistical mechanics [7–17]. However, in the last few years, a considerable effort has been spent to studies on thermo-statistical properties of two-parameter generalized bosonic and fermionic oscillator gas models which are invariant under the actions of some specific quantum groups. In particular, after the pioneering studies of Ubriaco [18–20] related with the high- and low-temperature behaviours of one-parameter deformed quantum group $SU_q(2)$-invariant bosonic and fermionic gases, two special kinds of generalizations of Ubriaco's works have recently been introduced in the literature [21–28]. These generalizations contain mainly two different quantum group bosonic and fermionic gas structures, whose particle algebras under consideration are invariant under the quantum groups $GL_{p,q}(2)$ with $p = q^*$ where $(p, q) \in C \times C$ and $SU_r(2)$ with $r = q_1 q_2^{-1}$ where $(q_1, q_2) \in R \times R$, respectively.

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Furthermore, it was recently shown in [20, 21] that both the one-parameter $SU_q(2)$- and its two-parameter generalization $SU_{q_1/q_2}(2)$-invariant boson models exhibit remarkably an anyonic type of behaviour in some critical values of the model deformation parameters $q_1$ and $q_2$. Therefore, these models constitute an alternative approach to study systems with fractional statistics. Such interesting results have motivated us to study other aspects of two-parameter deformed quantum group invariant boson systems.

In this paper, we aim to discuss both the mathematical and thermodynamical differences between the $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariant bosonic oscillator gases in [21, 22]. However, we should mention that a similar comparative study for fermionic versions of these $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariant algebras was carried out by [25].

The paper is organized as follows: First, we review some basic definitions and properties concerning the $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariant bosonic oscillator algebras, respectively. Then we analyze the Fock space representations of both of these two-parameter generalized bosonic algebras. In fact, such a consideration will provide essential algebraic differences between the $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariant boson models. We also discuss the high-temperature thermodynamical behaviours of these quantum group gases with $GL_{p,q}(2)$ and $SU_{q_1/q_2}(2)$ symmetries. Finally, we compare the mathematical and thermo-statistical properties of these quantum group invariant bosonic gas models, and give our conclusions.

## 2 Quantum Groups $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-Invariant Bosons

In this section, we recall the general properties of two special kinds of two-parameter deformed bosonic oscillator algebras invariant under the quantum groups $GL_{p,q}(2)$ with $p=q^*$ and $SU_r(2)$ with $r=q_1q_2^{-1}$, respectively [29–32]. Such two-parameter bosonic studies were also considered by several other authors [33–36].

The $(pq)$-deformed bosonic oscillator algebra generated by the bosonic field operators $\Phi_i$, $i=1,2$, is given by the following relations [22, 29, 30]:

$$\begin{aligned}
\Phi_1\Phi_1^* - pq\Phi_1^*\Phi_1 &= 1+(pq-1)\Phi_2^*\Phi_2,\\
\Phi_2\Phi_2^* - pq\Phi_2^*\Phi_2 &= 1,\\
\Phi_1\Phi_2 &= p^{-1}\Phi_2\Phi_1,\\
\Phi_1\Phi_2^* &= p\Phi_2^*\Phi_1,\\
\Phi_2\Phi_1^* &= q\Phi_1^*\Phi_2,\\
\Phi_1^*\Phi_2^* &= q\Phi_2^*\Phi_1^*,
\end{aligned} \tag{1}$$

where $p$ and $q$ are the complex deformation parameters such that $q=|q|\exp(i\theta)$ and $\theta$ is a phase. Note that taking hermitian conjugates of these relations implies $p=q^*$. We also give a relation $(p/q)=\exp(-2i\theta)$ which will be useful in the next steps of this study.

Under the linear transformation $\Phi_i' = \sum_{j=1}^{2} A_{ij}\Phi_j$, where the matrix $A\in GL_{p,q}(2)$, the relations given in (1) are invariant. The elements of the defining representation matrix $A=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ satisfy the following algebra [29–31]:

$$\begin{aligned}
&ab=qba, \qquad ac=pca, \qquad cd=qdc,\\
&bd=pdb, \qquad bc=(p/q)cb,\\
&ad-da=(q-p^{-1})bc,
\end{aligned}$$

$$D \equiv \det_{p,q}(A) = ad - qbc = ad - pcb,$$

where $p, q \in C/\{0\}$. In particular, setting $p = q$ one would obtain the relations of $SL_q(2)$.

On the other hand, the $SU_{q_1/q_2}(2)$-invariant bosonic oscillator algebra was first introduced in [32] during a realization of the most general quantum group invariant oscillator algebra. The two-parameter generalized bosonic quantum gas with $SU_{q_1/q_2}(2)$-symmetry is generated by the quantum group invariant bosonic $\Phi_i$ oscillators and defined by the following deformed commutation relations [32]:

$$\begin{aligned}
\Phi_1\Phi_1^* - q_1^2\Phi_1^*\Phi_1 &= q_2^{2M}, \\
\Phi_2\Phi_2^* - q_1^2\Phi_2^*\Phi_2 &= q_2^{2M} + (q_1^2 - q_2^2)\Phi_1^*\Phi_1, \\
\Phi_1\Phi_2 &= q_1q_2^{-1}\Phi_2\Phi_1, \\
\Phi_1^*\Phi_2^* &= q_2q_1^{-1}\Phi_2^*\Phi_1^*, \\
\Phi_1\Phi_2^* &= q_1q_2\Phi_2^*\Phi_1,
\end{aligned} \tag{2}$$

where $M$ is the total boson number operator, and $q_1$, $q_2$ are the real independent deformation parameters. Hereafter, we will consider $0 < q_1 < \infty$ and $0 < q_2 < \infty$. One can check that the two-dimensional bosonic oscillator algebra in (2) shows $SU_{q_1/q_2}(2)$-symmetry. Our deformed bosonic algebra is invariant under the following transformation:

$$\begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} = T\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} a & -q_1q_2^{-1}b^* \\ b & a^* \end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{3}$$

such that $T$ is the transformation matrix and $T \in SU_r(2)$ with $r = q_1/q_2$. The elements of matrix $T$ satisfy the following equations:

$$\begin{aligned}
&ab = q_1q_2^{-1}ba, \qquad ab^* = q_1q_2^{-1}b^*a, \qquad bb^* = b^*b, \\
&aa^* + q_1^2q_2^{-2}b^*b = 1, \qquad a^*a + bb^* = 1.
\end{aligned} \tag{4}$$

If we rewrite all relations in (2) for the transformed ones, one can readily see that our system remains unchanged. It is important to remark that the matrix elements of $T$ effectively involve a single parameter $r = q_1/q_2$ and are assumed to commute with $\Phi_1$, $\Phi_2$, $\Phi_1^*$, $\Phi_2^*$.

We also note that the $SU_{q_1/q_2}(2)$-invariant boson algebra in (2) has some interesting limiting cases via the deformation parameters $q_1$ and $q_2$, which will be discussed in final section.

## 3 Fock Space Representations for $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-Invariant Boson Algebras

In this section, we construct the Fock space representations of the two-parameter deformed bosonic oscillator algebras in (1) and (2). For the $GL_{p,q}(2)$-invariant boson algebra in (1), we introduce the Fock basis $|d_1, d_2\rangle$ and the representations of the operators $\Phi_i$ and $\Phi_i^\star$, $i = 1, 2$, can be calculated as follows:

$$\Phi_1|d_1, d_2\rangle = |p|^{d_2}\sqrt{\left(\frac{1-(qp)^{d_1}}{1-qp}\right)}|d_1 - 1, d_2\rangle,$$

$$\begin{aligned}
\Phi_1^{\star}|d_1,d_2\rangle &= |q|^{d_2}\sqrt{\left(\frac{1-(qp)^{d_1+1}}{1-qp}\right)}|d_1+1,d_2\rangle,\\
\Phi_2|d_1,d_2\rangle &= \sqrt{\left(\frac{1-(qp)^{d_2}}{1-qp}\right)}|d_1,d_2-1\rangle,\\
\Phi_2^{\star}|d_1,d_2\rangle &= \sqrt{\left(\frac{1-(qp)^{d_2+1}}{1-qp}\right)}|d_1,d_2+1\rangle,\\
\Phi_1^{*}\Phi_1|d_1,d_2\rangle &= (qp)^{d_2}\left(\frac{1-(qp)^{d_1}}{1-qp}\right)|d_1,d_2\rangle,\\
\Phi_2^{*}\Phi_2|d_1,d_2\rangle &= \left(\frac{1-(qp)^{d_2}}{1-qp}\right)|d_1,d_2\rangle,\\
\Phi_1|0,0\rangle &= 0, \qquad \Phi_2|0,0\rangle = 0,
\end{aligned} \tag{5}$$

where $p=q^*$, $(p,q)\in C\times C$ and $d_1,d_2=0,1,2,\ldots$. From these representations, the total deformed number operator for these $(pq)$-deformed oscillators is

$$\Phi_1^*\Phi_1+\Phi_2^*\Phi_2=[D_1+D_2]=[D], \tag{6}$$

whose spectrum is given by

$$[d]=\frac{1-(qp)^d}{1-qp}. \tag{7}$$

From the bosonic vacuum condition $\Phi_i|0,0\rangle=0$, one can construct an orthonormal $(pq)$-boson state $|d_1,d_2\rangle$:

$$|d_1,d_2\rangle=\frac{1}{\sqrt{[d_1]![d_2]!}}(\Phi_2^{\star})^{d_2}(\Phi_1^{\star})^{d_1}|0,0\rangle, \tag{8}$$

where $[d_i]$ is defined in (7). Here, we should emphasize that $[d]$ can not vanish. Since $qp=|q|^2$ and $|q|$ must be real and positive. Hence, the Fock states defined in (8) are defined for all integer values of $d$. Such a result is in contrast for theories with complex deformation parameters. Since, for complex $q$, the Fock space gets undefined for the reason that $[d]=0$ in some cases as pointed out by [10], and hence leads to the breaking up of Fock space into disjoint subspaces. Thus, in spite of the two complex deformation parameters $q$ and $p$ in the present $GL_{p,q}(2)$-invariant model, such a problem does not arise for (7) due to the condition $p=q^*$.

On the other hand, for the $SU_{q_1/q_2}(2)$-invariant boson algebra in (2), let $|m_1,m_2\rangle$ be the Fock space basis and the ground state satisfies $\Phi_i|0,0\rangle=0$, $i=1,2$. The actions of the deformed bosonic oscillators $\Phi_i$ and $\Phi_i^\star$ on the states are found as follows:

$$\begin{aligned}
\Phi_1|m_1,m_2\rangle &= q_2^{m_2}\sqrt{\left(\frac{q_1^{2m_1}-q_2^{2m_1}}{q_1^2-q_2^2}\right)}|m_1-1,m_2\rangle,\\
\Phi_1^\star|m_1,m_2\rangle &= q_2^{m_2}\sqrt{\left(\frac{q_1^{2(m_1+1)}-q_2^{2(m_1+1)}}{q_1^2-q_2^2}\right)}|m_1+1,m_2\rangle,\\
\Phi_2|m_1,m_2\rangle &= q_1^{m_1}\sqrt{\left(\frac{q_1^{2m_2}-q_2^{2m_2}}{q_1^2-q_2^2}\right)}|m_1,m_2-1\rangle,\\
\Phi_2^\star|m_1,m_2\rangle &= q_1^{m_1}\sqrt{\left(\frac{q_1^{2(m_2+1)}-q_2^{2(m_2+1)}}{q_1^2-q_2^2}\right)}|m_1,m_2+1\rangle,\\
\Phi_1^*\Phi_1|m_1,m_2\rangle &= q_2^{2m_2}\left(\frac{q_1^{2m_1}-q_2^{2m_1}}{q_1^2-q_2^2}\right)|m_1,m_2\rangle,\\
\Phi_2^*\Phi_2|m_1,m_2\rangle &= q_1^{2m_1}\left(\frac{q_1^{2m_2}-q_2^{2m_2}}{q_1^2-q_2^2}\right)|m_1,m_2\rangle,
\end{aligned} \tag{9}$$

where $q_1 \neq q_2$, $(q_1,q_2) \in R \times R$ and $m_1, m_2 = 0, 1, 2, \ldots$. From these representations, the total deformed boson number operator for these quantum group invariant $(q_1,q_2)$-deformed oscillators can be deduced as

$$\Phi_1^*\Phi_1 + \Phi_2^*\Phi_2 = [M_1 + M_2] = [M], \tag{10}$$

whose spectrum is defined by the following generalized Fibonacci basic integer $[m]$:

$$[m] = \frac{q_1^{2m} - q_2^{2m}}{q_1^2 - q_2^2}, \tag{11}$$

which is a two-parameter generalization of the usual $q$-numbers. Due to this fact, the $(q_1,q_2)$-deformed bosonic oscillators in (2) are called Fibonacci oscillators, and the constants $q_1$ and $q_2$ are also called as the parameters of the Fibonacci basic integers [32]. All of thermodynamical and statistical functions for the present $SU_{q_1/q_2}(2)$-invariant boson gas model are given in terms of this generalized $q$-numbers, the generalized Fibonacci basic integer $[m]$ will be of crucial importance in this model.

The state $|m_1,m_2\rangle$ is obtained by applying the Fibonacci creation operators $\Phi^*$ on the ground state successively as

$$|m_1,m_2\rangle = \frac{1}{\sqrt{[m_1]![m_2]!}}(\Phi_2^\star)^{m_2}(\Phi_1^\star)^{m_1}|0,0\rangle, \tag{12}$$

where the generalized Fibonacci basic integer $[m_i]$ is defined in (11).

## 4 $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-Invariant Boson Models

In this section, we review some of the properties of two special kinds of two-parameter generalized bosonic gases whose particle algebras are invariant under the quantum groups

$GL_{p,q}(2)$ and $SU_{q_1/q_2}(2)$, respectively [21, 22]. Such a discussion will be useful for comparison of high-temperature thermodynamical behaviours of these quantum group invariant bosonic gases in the next section. For such an aim, we first summarize some of the defining relations of the $GL_{p,q}(2)$-invariant boson model with $p = q^*$ described by the following Hamiltonian in terms of $GL_{p,q}(2)$-generators [22]:

$$H = \sum_k \varepsilon_k(\widehat{D}_{1,k} + \widehat{D}_{2,k}), \tag{13}$$

where the operators $\widehat{D}_{1,k}$ and $\widehat{D}_{2,k}$ are given by

$$\widehat{D}_{1,k} = \Phi^*_{1,k}\Phi_{1,k}, \qquad \widehat{D}_{2,k} = \Phi^*_{2,k}\Phi_{2,k}, \tag{14}$$

$\varepsilon_k$ is the spectrum of energy, $k = 0, 1, 2, \dots$. The following relations are satisfied by the operators $\widehat{D}_1$, $\widehat{D}_2$ for a given $k$:

$$\widehat{D}_2\Phi_1 - \Phi_1\widehat{D}_2 = 0, \qquad q\widehat{D}_1\Phi_2 - p^{-1}\Phi_2\widehat{D}_1 = 0. \tag{15}$$

In order to calculate the thermodynamical properties of the Hamiltonian in (13), the new representation was proposed for the $GL_{p,q}(2)$-invariant bosonic $\Phi_i$ oscillators in terms of the ordinary bosonic oscillators $\phi_{i,k}$ and $\phi^*_{i,k}$ for a given $k$ as follows [22]:

$$\begin{aligned} &\Phi_2 = (\phi_2^*)^{-1}[D_2], \qquad \Phi_2^* = \phi_2^*, \\ &\Phi_1 = (\phi_1^*)^{-1}[D_1]p^{D_2}, \qquad \Phi_1^* = \phi_1^* q^{D_2}, \end{aligned} \tag{16}$$

where $p = q^*$, $(p, q) \in C \times C$. Using these representations, one can rewrite the Hamiltonian in (13) as follows:

$$H = \sum_k \varepsilon_k[D_{1,k} + D_{2,k}], \tag{17}$$

where $D_{i,k} = \phi^*_{i,k}\phi_{i,k}$ and the bracket $[x]$ is defined in (7). Note that when compared with the original Hamiltonian in (13), this representation leads to an interacting Hamiltonian [22].

For comparison, we now summarize some of the defining relations of the $SU_{q_1/q_2}(2)$-invariant boson model described by the following Hamiltonian in terms of $SU_{q_1/q_2}(2)$-generators [21]:

$$H = \sum_k \varepsilon_k(\widetilde{M}_{1,k} + \widetilde{M}_{2,k}), \tag{18}$$

where the deformed Fibonacci number operators $\widetilde{M}_{1,k}$ and $\widetilde{M}_{2,k}$ are defined by

$$\widetilde{M}_{1,k} = \Phi^*_{1,k}\Phi_{1,k}, \qquad \widetilde{M}_{2,k} = \Phi^*_{2,k}\Phi_{2,k}, \tag{19}$$

$\varepsilon_k$ is the spectrum of energy, $k = 0, 1, 2, \dots$, and $[\Phi^*_{i,k}, \Phi_{j,k'}] = 0$ for $k \neq k'$. These operators also satisfy the following relations for a given $k$:

$$\widetilde{M}_2\Phi_1 - q_1^{-2}\Phi_1\widetilde{M}_2 = 0, \qquad \widetilde{M}_1\Phi_2 - q_2^{-2}\Phi_2\widetilde{M}_1 = 0. \tag{20}$$

In order to express a new representation for Fibonacci oscillators $\Phi_i$ in terms of the undeformed boson operators $\phi_i$ and $\phi_i^*$ satisfying the relations $\phi_i\phi_j^* - \phi_j^*\phi_i = \delta_{ij}$, $\phi_i\phi_j - \phi_j\phi_i = 0$,

$M_i = \phi_i^* \phi_i$, $i, j = 1, 2$, we propose the following representations for a given $k$ [21]:

$$\Phi_1 = (\phi_1^*)^{-1}[M_1] q_2^{M_2}, \qquad \Phi_1^* = \phi_1^* q_2^{M_2}, \tag{21}$$

$$\Phi_2 = (\phi_2^*)^{-1}[M_2] q_1^{M_1}, \qquad \Phi_2^* = \phi_2^* q_1^{M_1}, \tag{22}$$

where $q_1 \neq q_2$, $(q_1, q_2) \in R \times R$. By means of these representations, we are able to rewrite the original Hamiltonian in (18) as

$$H = \sum_k \varepsilon_k [M_{1,k} + M_{2,k}], \tag{23}$$

where the spectrum of the bracket $[M_1 + M_2]$ is given by the generalized Fibonacci basic integer $[m_i]$ in (11). It is important to note that when we compare this new Hamiltonian with the original Hamiltonian in (18), the representations in (21) and (22) bring about an interacting Hamiltonian for the system of two different kinds of bosonic particle families. This results from the $SU_{q_1/q_2}(2)$-symmetry of the system. Also, such an interaction is fixed by the deformation parameters $q_1$ and $q_2$. The non-interacting system generated by free bosonic particles can obviously be obtained in the limit $q_1 = q_2 = 1$. The differences between the above $GL_{p,q}(2)$ and $SU_{q_1/q_2}(2)$-invariant boson models will be discussed in final section.

## 5 High-Temperature Thermodynamical Behaviours of the $SU_{q_1/q_2}(2)$- and $GL_{p,q}(2)$-Invariant Boson Gases

For comparison, we need to mention some important high-temperature thermodynamical properties of the $SU_{q_1/q_2}(2)$ and $GL_{p,q}(2)$-invariant boson gas models, respectively [21, 22]. Besides, we shall find the internal energies and the specific heats for these models in terms of the deformation parameters $(q_1, q_2)$ and $(pq)$, respectively. In particular, we shall discuss the effect of the deformation parameters on the specific heats for these models.

For a three-dimensional investigation of the high-temperature behaviour of the $SU_{q_1/q_2}(2)$-invariant boson model [21], we use its $SU_{q_1/q_2}(2)$-invariant model Hamiltonian in (23) in order to obtain the grand partition function in the limit $z = \exp(\beta\mu) \ll 1$ as

$$\ln Z^{(q_1,q_2)} = \frac{V}{\lambda^3}(2z + 4z^2 \xi(q_1, q_2) + \cdots), \tag{24}$$

where the thermal wavelength is $\lambda = \sqrt{(2\pi\hbar^2/mkT)}$ and the $(q_1, q_2)$-deformed function $\xi(q_1, q_2)$ is

$$\xi(q_1, q_2) = \frac{1}{4}\left[\frac{3}{(q_1^2 + q_2^2)^{3/2}} - \frac{1}{\sqrt{2}}\right]. \tag{25}$$

From (24), we can obtain all the thermodynamical functions of the $SU_{q_1/q_2}(2)$-invariant boson model in terms of some functions of the real independent deformation parameters $q_1$ and $q_2$. For instance, the average number of particles $\langle N\rangle$, the internal energy $U$ and the specific heat $C_V$ as functions of $q_1$ and $q_2$ are derived as follows:

$$\langle N\rangle = \frac{V}{\lambda^3}(2z + 8z^2 \xi(q_1, q_2) + \cdots), \tag{26}$$

$$U = \frac{3}{2}\frac{\langle N\rangle}{\beta}\left(1 - \frac{\lambda^3 \xi(q_1, q_2)\langle N\rangle}{V} + \cdots\right), \tag{27}$$

$$C_V = \frac{3}{2}k\langle N\rangle\left(1 + \frac{\lambda^3\xi(q_1,q_2)\langle N\rangle}{2V} + \cdots\right). \tag{28}$$

Also, the specific heat in (28) can be rewritten up to the second order in $T$ as follows:

$$\frac{C_V}{k\langle N\rangle} = \frac{3}{2}\left(1 + \frac{1}{2}\xi(q_1,q_2)\tilde{g}_{3/2}(1,q_1,q_2)\left(\frac{T_c(q_1,q_2)}{T}\right)^{3/2}\right), \tag{29}$$

where the critical temperature $T_c(q_1,q_2)$ and the $(q_1,q_2)$-deformed function $\tilde{g}_{3/2}(1,q_1,q_2)$ are defined as

$$T_c(q_1,q_2) = \frac{2\pi\hbar^2/mk}{[(\frac{V}{\langle N\rangle})\tilde{g}_{3/2}(1,q_1,q_2)]^{2/3}}, \tag{30}$$

$$\tilde{g}_{3/2}(1,q_1,q_2) = \sum_{n=1}^{\infty}\frac{1}{[n]^{3/2}}, \tag{31}$$

where the generalized Fibonacci basic integer $[n]$ is given by (11). For high temperatures, in Figs. 1 and 2, we show the plots of the specific heat $C_V/k\langle N\rangle$ as a function of $T/T_c(q_1,q_2)$ for several values of the deformation parameters $q_1$ and $q_2$.

Furthermore, in three-dimensional space, the equation of state is derived as a virial expansion depending on the deformation parameters $q_1$ and $q_2$ as

$$PV = kT\langle N\rangle\left[1 - \xi(q_1,q_2)\lambda^3\frac{\langle N\rangle}{V} + \cdots\right], \tag{32}$$

where the second virial coefficient $B_2(q_1,q_2)$ is

$$B_2(q_1,q_2) = \xi(q_1,q_2)\lambda^3. \tag{33}$$

Obviously, the sign of the second virial coefficient $B_2(q_1,q_2)$ depends on the real independent deformation parameters $q_1$ and $q_2$. The $(q_1,q_2)$-deformed function $\xi(q_1,q_2)$ in (25)

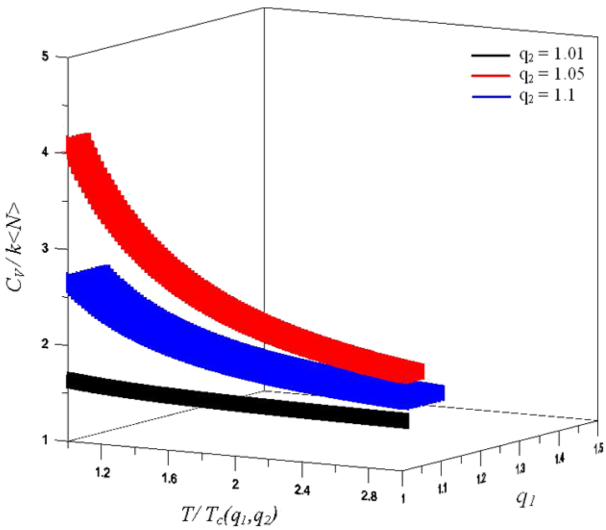


**Fig. 1** The specific heat $C_V/k\langle N\rangle$ as a function of $T/T_c(q_1,q_2)$ for various values of the deformation parameters $q_2$ and $1 \le q_1 \le 1.5$

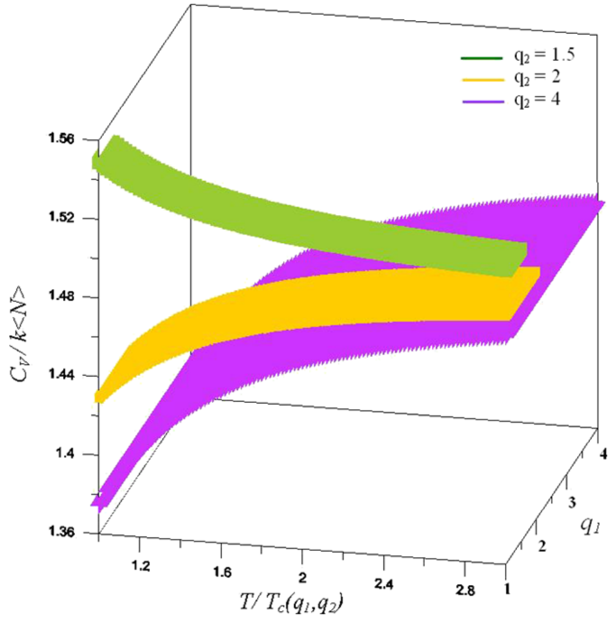


**Fig. 2** The specific heat $C_V/k\langle N\rangle$ as a function of $T/T_c(q_1,q_2)$ for various values of the deformation parameters $q_2$ and $1 \leq q_1 \leq 4$

vanishes at $(q_1^2+q_2^2)\approx 2.62$ which gives an ideal gas result up to the second virial coefficient. A free boson gas result $\xi(1,1)=2^{-7/2}$ can be obtained in the limit $q_1=q_2=1$, whereas a free fermion gas result $\xi(q_1,q_2)=-2^{-7/2}$ is reached at $(q_1^2+q_2^2)\approx 4.16$ [24].

On the other hand, from a two-dimensional investigation for such a $SU_{q_1/q_2}(2)$-invariant boson gas [21], the equation of state is found as

$$PA = kT\langle N\rangle\left[1-\zeta(q_1,q_2)\lambda^2\frac{\langle N\rangle}{A}+\cdots\right], \tag{34}$$

where the second virial coefficient $B_2(q_1,q_2)$ becomes

$$B_2(q_1,q_2)=\zeta(q_1,q_2)\lambda^2. \tag{35}$$

where the $(q_1,q_2)$-deformed function $\zeta(q_1,q_2)$ is

$$\zeta(q_1,q_2)=\frac{1}{4}\left[\frac{3}{(q_1^2+q_2^2)}-1\right]. \tag{36}$$

This function vanishes at $(q_1^2+q_2^2)\approx 3.0$ such that it gives an ideal gas result up to the second virial coefficient. A free boson gas result $\zeta(1,1)=2^{-3}$ can be recovered in the limit $q_1=q_2=1$, whereas a free fermion gas result $\zeta(q_1,q_2)=-2^{-3}$ can be obtained at $(q_1^2+q_2^2)\approx 6.0$ [24]. Therefore, the $SU_{q_1/q_2}(2)$-invariant boson system behaves both as a bosonic and a repulsive system in the region $(q_1^2+q_2^2)<3.0$, whereas it behaves both a fermion-like and an attractive system in the region $(q_1^2+q_2^2)>3.0$. We should emphasize that a similar behaviour can also be deduced in the three dimensional investigation of this model given in (32) and (33). Hence, the real independent deformation parameters $q_1$ and $q_2$ interpolate between bosonic and fermion-like behaviours of the system. These parameters are responsible for the behaviour of the present two-parameter $SU_{q_1/q_2}(2)$-invariant boson model. We should mention for all equations of this model that the limit $q_2 = 1$ gives the same results derived in [20] for the one-parameter $SU_q(2)$-invariant boson gas model.

On the other hand, for comparison, it is useful to summarize some of the results of the $GL_{p,q}(2)$-invariant boson model [22] in a slightly different form in order to find the internal energy and the specific heat of the model. Using the $GL_{p,q}(2)$-invariant Hamiltonian in (17), the grand partition function in the high-temperature limit is

$$\ln Z^{(pq)} = \frac{V}{\lambda^3}(2z + 4z^2\delta(p,q) + \cdots), \tag{37}$$

where the function $\delta(p,q)$ is

$$\delta(p,q) = \frac{1}{4}\left[\frac{3}{(1+pq)^{3/2}} - \frac{1}{\sqrt{2}}\right], \tag{38}$$

where the complex deformation parameters $p$ and $q$ should satisfy the constraint $p = q^*$ due to consistent hermitian conjugation properties of the $GL_{p,q}(2)$-invariant boson algebra in (1) as has been shown in [22, 29–31, 37]. In this case, one can obtain the average number of particles $\langle N\rangle$, the internal energy $U$ and the specific heat $C_V$ for high temperatures as follows:

$$\langle N\rangle = \frac{V}{\lambda^3}\big(2z + 8z^2\delta(p,q) + \cdots\big), \tag{39}$$

$$U = \frac{3}{2}\frac{\langle N\rangle}{\beta}\left(1 - \frac{\lambda^3\delta(p,q)\langle N\rangle}{V} + \cdots\right), \tag{40}$$

$$C_V = \frac{3}{2}k\langle N\rangle\left(1 + \frac{\lambda^3\delta(p,q)\langle N\rangle}{2V} + \cdots\right). \tag{41}$$

For the $GL_{p,q}(2)$-invariant boson gas, the specific heat in (41) can also be rewritten up to the second order in $T$ as follows:

$$\frac{C_V}{k\langle N\rangle} = \frac{3}{2}\left(1 + \frac{1}{2}\delta(p,q)g_{3/2}(1,pq)\left(\frac{T_c(1,pq)}{T}\right)^{3/2}\right), \tag{42}$$

where the critical temperature $T_c(1,pq)$ and the function $g_{3/2}(1,pq)$ are defined as

$$T_c(1,pq) = \frac{2\pi\hbar^2/mk}{[(\frac{V}{\langle N\rangle})g_{3/2}(1,pq)]^{2/3}}, \tag{43}$$

$$g_{3/2}(1,pq) = \sum_{d=1}^{\infty}\frac{1}{[d]^{3/2}}, \tag{44}$$

where $[d]$ is given by (7). For high temperatures, in Fig. 3, we show the plot of the specific heat $C_V/k\langle N\rangle$ as a function of $T/T_c(1,pq)$ for several values of the deformation parameter $pq$.

Furthermore, in three-dimensional space, the equation of state for this model was found [22] as

$$PV = kT\langle N\rangle(1 - \delta(p,q))\left[\lambda^3\frac{\langle N\rangle}{V} + \cdots\right]. \tag{45}$$

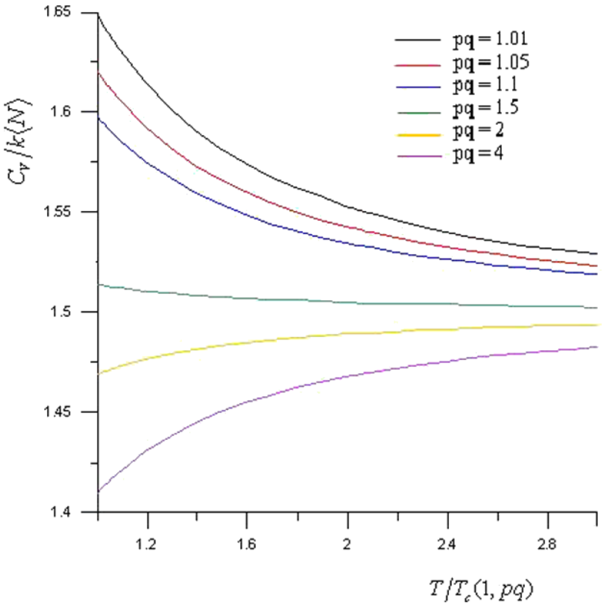


**Fig. 3** The specific heat $C_V/k\langle N\rangle$ as a function of $T/T_c(1, pq)$ for various values of the deformation parameter $pq$

Moreover, from a two-dimensional investigation for such $GL_{p,q}(2)$-invariant model [22], the equation of state was derived as

$$PA = kT\langle N\rangle\left[1 - \eta(p,q)\lambda^2\frac{\langle N\rangle}{A} + \cdots\right], \tag{46}$$

where the function $\eta(p, q)$ is defined by

$$\eta(p,q) = \frac{2 - pq}{4(1 + pq)}, \tag{47}$$

with the condition $p = q^*$ for the same reason discussed above. Now we can compare these results with the $SU_{q_1/q_2}(2)$-invariant boson gas results.

## 6 Discussion and Conclusions

In this paper, we have discussed the quantum group invariant bosonic gases through their $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariances. Particularly, we have investigated the Fock space properties of both of the bosonic oscillator gases with two different symmetries: $GL_{p,q}(2)$ and $SU_{q_1/q_2}(2)$. We have also established the nature of the two different basic numbers which followed from these two-parameter boson algebras. We have studied the defining commutation relations for the two-parameter generalized $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariant bosonic algebras.

Moreover, we discussed the high-temperature thermodynamical behaviours of the quantum group bosonic gases with two distinct symmetries: $GL_{p,q}(2)$ and $SU_{q_1/q_2}(2)$. By means of the $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariant bosonic Hamiltonians, we calculated several thermodynamical functions via the bosonic grand partition functions for the two systems. These functions are expressed in terms of the model deformation parameters $(q_1, q_2)$ and $(pq)$, respectively. For instance, the critical temperatures and the internal energies for both models are derived for high temperatures. Subsequently, the specific heats for both systems are obtained in the high-temperature limit. We then focused on the effects of the deformation parameters on these results. All the thermodynamical functions obtained for both models become their respective free boson gas functions with two different kinds of bosonic particles in the limits ($\theta = 0$, $|q| = 1$) and ($q_1 = q_2 = 1$), respectively. As shown in Figs. 1 and 2, when the second deformation parameter $q_2$ increases, the specific heat of the $SU_{q_1/q_2}(2)$-invariant boson gas decreases, and it becomes approximately constant for the temperatures $T/T_c(q_1, q_2) \geq 2.4$. On the other hand, as shown in Fig. 3, the specific heat of the $GL_{p,q}(2)$-invariant boson model decreases as the deformation parameter $pq$ is increased, and in this sense, it interestingly shows a reminiscent behaviour as in the case of the one-parameter deformed $SU_q(2)$-invariant boson gas model in [20].

The two-parameter realizations of the quantum group invariant bosonic oscillator algebras in (1) and (2) offer crucial algebraic and representative differences: The quantum group $GL_{p,q}(2)$-invariant bosonic algebra constitutes essentially an invariance under the action of a unitarized form of this quantum group with $p = q^*$, named $U_{q,q^*}(gl(2))$ [37], whereas the quantum group $SU_{q_1/q_2}(2)$-invariant bosonic algebra exhibits an invariance under a special case of the quantum group $GL_r(2)$ with $r = q_1 q_2^{-1}$, $(q_1, q_2) \in R \times R$. Although, the quantum group $GL_{p,q}(2)$-bosons contain the quantum group $SU_{q_1/q_2}(2)$-bosons, it is not possible to make a rescaling or a transformation from $GL_{p,q}(2)$-invariant bosonic algebra to $SU_{q_1/q_2}(2)$-invariant one. Therefore, we conclude that these two quantum group invariant bosons have completely independent algebraical structures, and hence their all of thermodynamical properties are originally different.

As is proved in Sect. 2, the invariance quantum group of the $(q_1, q_2)$-deformed Fibonacci oscillator algebra in (2) is the $SU_r(2)$ which has effectively a single real deformation parameter $r = q_1/q_2$, whereas the $(pq)$-deformed boson algebra with the constraint $p = q^*$ in (1) is invariant under the quantum group $GL_{p,q}(2)$, which has two complex deformation parameters $p$ and $q$. However, from the differential calculus on $GL_{p,q}(2)$ and its Lie algebra studied in [29, 30], we see that the Lie algebra of the two-parameter deformation of $GL(2)$ is essentially a one-parameter deformation of the classical Lie algebra $gl(2)$, which results from the fact that both the algebra of generators and the structure of the $R$-matrix display the algebra deformation by the parameter $X = pq$.

The quantum group $GL_{p,q}(2)$-invariant bosonic system gives the ordinary $SU(2)$-invariant bosons in the limit $\theta = 0$ and $|q| = 1$. When we take the limit $\theta = 0$, the usual $SU_q(2)$-invariant bosonic algebra can also be recovered [38]. Whenever we discuss any of the limiting cases of the $GL_{p,q}(2)$-invariant bosonic algebra, we have to consider the constraint $p = q^*$ and the relation $(p/q) = \exp(-2i\theta)$. Thus we must take into account the phase angle $\theta$ between the complex deformation parameters $p$ and $q$. They are no longer independent but they mainly are dependent on each other. This was not considered in [22]. Therefore, all the thermo-statistical characteristics of the $GL_{p,q}(2)$-invariant boson model depend on the parameters $q$ and $p = q^*$, and thus the phase angle $\theta$. For this model, all the thermodynamical relations given in (37)–(47) coincide merely with the one-parameter $SU_q(2)$-boson gas results [20] in the limit $\theta = 0$. However, for the $GL_{p,q}(2)$-invariant boson model, it is not only enough to remark that the limit $p = q$ gives the $SU_q(2)$-boson gas results [20], but it is also necessary to consider the phase angle $\theta$ between the complex deformation parameters $p$ and $q$.

On the other hand, the quantum group $SU_{q_1/q_2}(2)$-invariant bosons give the ordinary $SU(2)$-invariant bosons in the limit $q_1 = q_2 = 1$. They also reveal the one-parameter deformed bosonic algebra invariant under the quantum group $SU_q(2)$ in the limit $q_2 = 1$ [38].

When we take the limit $q_1 = q_2$, we find the two-dimensional bosonic Newton oscillator algebra [39, 40] invariant under the undeformed group $SU(2)$.

Moreover, we would like to discuss other effects of the real independent deformation parameters $q_1$ and $q_2$ on the algebraic structure of a system of the $SU_{q_1/q_2}(2)$-invariant bosonic oscillators. These deformation parameters play important roles for the system under consideration. Not only do they constitute a quantum deformation of the classical symmetry group of the system, but also they bring about an interaction via the $SU_{q_1/q_2}(2)$-symmetry between two bosonic particle families. However, this does not mean that deformation is equivalent to an interaction. Since quantum deformation may not necessarily be the same as an interaction between the particles of the two families. The interaction conjectured in our $SU_{q_1/q_2}(2)$-boson model comes from the unitary quantum group symmetry of the system presented by a deformed Hamiltonian in (23). This remark gives also a main difference between the $SU_{q_1/q_2}(2)$-bosons and the so called $q$-bosons [5, 6]. Strictly speaking, the $q$-bosons does not have a covariance under a quantum group structure.

With the light of the above discussion, we may interpret our model as containing two different kinds of bosonic oscillator families which interact with each other via the deformation parameters $q_1$ and $q_2$ fixed by the quantum group $SU_{q_1/q_2}(2)$-symmetry. But these two bosonic families do not interact among themselves. Therefore, in some sense, the entire behavior of the system is characterized by the model parameters $q_1$ and $q_2$. When we take the limit $q_1 = q_2 = 1$, the non-interacting system with two different kinds of ordinary bosons can be recovered. In the limit $q_2 = 1$, these two bosonic particle families are also interacting via the deformation parameter $q_1$, but in this case, one of the bosonic oscillator families does not have the same physical properties with the other.

It deserves to mention that, for high temperatures, the present $SU_{q_1/q_2}(2)$-invariant boson gas [21] behaves as a fermion gas at the value of $(q_1^2 + q_2^2) \approx 4.16$ and it also shows the Bose-Einstein condensation [28] for low temperatures in the interval $q_2 > q_1 > 0$. Obviously, the results for the free boson gas can be found in the limit $q_1 = q_2 = 1$.

As is discussed in Sect. 5, the thermo-statistical results for $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariant boson models having the model Hamiltonians in (17) and (23) are obviously different, since the nature of the two-parameter deformations of the quantum group invariant bosonic oscillator algebras for these models is quite different on the algebraic basis. The differences between these two-parameter realizations result not only from the defining commutation relations of both of the bosonic oscillator algebras in (1) and (2), but also from the Fock space representation properties of the two algebras. Actually, we should emphasize that a unitarization of the quantum group $GL_{p,q}(2)$ with $p \neq q^*$, called $U_{p,q}(2)$, has recently been introduced in [41], where the representations of such an algebra were constructed and the relationships of these representations to $q$-oscillators were also discussed. In the limit $p = q^*$, the new algebra $U_{p,q}(2)$ in [41] coincides with the algebra $U_{q,q^*}(gl(2))$ [37], which is an invariance quantum group of the study of [22] discussed in Sect. 2. Therefore, from such an algebraic observation, we remark that the actual two-parameter generalization of the results of the one-parameter deformed $SU_q(2)$-invariant boson model [20] is the one with the $SU_{q_1/q_2}(2)$-symmetry. The reason behind this conclusion is that the $GL_{p,q}(2)$-invariant boson algebra in (1) is consistent only under the constraint $p = q^*$.

To conclude, our results indicate that the algebras and therefore thermostatistics of the present two-parameter $GL_{p,q}(2)$- and $SU_{q_1/q_2}(2)$-invariant boson models are clearly different. In addition, they deserve more future studies. Some open problems are thermo-statistical properties of the $U_{p,q}(2)$-invariant boson gas model with $p \neq q^*$, and the consequences of imposing two-parameter quantum group symmetries in the formalism of nonextensive quantum statistical mechanics in which the main focus would be the roles of real independent deformation parameters $q_1$ and $q_2$ of the $SU_{q_1/q_2}(2)$-invariant boson model. Furthermore, it would be interesting to investigate the algebraic and statistical consequences of the $SU_{q_1/q_2}(2)$-invariant boson model when the deformation parameter $q_1/q_2$ is a root of unity. We hope that these problems will be addressed in the near future.

**Acknowledgements** This work is supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under the research project number 106T593. Also, the author would like to thank D. Irk and E. Arslan for their help in preparing the three-dimensional plots.

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